数理解析研究所講究録
388巻 1980年 183-204



# 広田氏の Bilinear Equations について

京大 数理研 佐藤 幹夫
琉球大 理 毛織 泰子

ここでは主に Korteweg-de Vries (KdV) 方程式, modified Korteweg-de Vries (MKdV) 方程式及び Nonlinear Schrödinger (NLS) 方程式を考える。

$\Delta(p_1, \cdots, p_r)$ で差積: $\prod_{1\le i<j\le r}(p_i - p_j)$ を表わす。このとき $\Delta(p_1^2, \cdots, p_r^2) = \prod_{i<j}(p_i^2 - p_j^2)$ である。

[KdV, MKdV]

$t = (t_0, t_1, t_2, \cdots)$ において, $t_0, t_1, t_2, \cdots$ の weight を各々 $1, 3, 5, \cdots$ と定める。$x = (x_0, x_1, x_2, \cdots)$ についても同様とする。

$p_1, \cdots, p_r$ の函数 $F(p_1, \cdots, p_r)$ の totally $\substack{\text{even}\\ \text{odd}}$ part を次のように定義する:

(1) $$\text{totally even part of } F(p_1, \cdots, p_r) \underset{\text{def.}}{=} \frac{1}{2^r} \sum_{\epsilon_1, \cdots, \epsilon_r = \pm 1}^{2^r} F(\epsilon_1 p_1, \cdots, \epsilon_r p_r).$$

(2) $$\text{totally odd part of } F(p_1, \cdots, p_r) \underset{\text{def.}}{=} \frac{1}{2^r} \sum_{\epsilon_1, \cdots, \epsilon_r = \pm 1}^{2^r} (-)^{\#} F(\epsilon_1 p_1, \cdots, \epsilon_r p_r),$$
$$\# \underset{\text{def.}}{=} \#\{ j \mid \epsilon_j = -1,\ 1 \le j \le r \}.$$

(3) $$\Delta(p_1^2, \cdots, p_r^2)\, \Phi^{\pm}(t; p_1, \cdots, p_r)$$
$$\underset{\text{def.}}{=} \text{totally } \substack{\text{even}\\ \text{odd}} \text{ part of } \Delta(p_1, \cdots, p_r)\, e^{\frac{t_0}{2}(p_1 + \cdots + p_r) + \frac{t_1}{2}(p_1^3 + \cdots + p_r^3) + \frac{t_2}{2}(p_1^5 + \cdots + p_r^5) + \cdots}.$$

(4) $$2^r\cdot\Delta(p_1^2,\cdots,p_r^2)\cdot\left(\Phi^-(t;p_1,\cdots,p_r)\right)^2 \overset{\mathrm{def.}}{=} \sum_{\substack{l_1>\cdots>l_r>0\\ \mathrm{even}}} a_{l_1\cdots l_r}(t)\cdot\det\begin{bmatrix} p_1^{l_1} & \cdots & p_1^{l_r}\\ \vdots & \ddots & \vdots\\ p_r^{l_1} & \cdots & p_r^{l_r}\end{bmatrix}.$$

(5) $$2^r\cdot\Delta(p_1^2,\cdots,p_r^2)\cdot\Phi^+(t;p_1,\cdots,p_r)\,\Phi^-(t;p_1,\cdots,p_r) \overset{\mathrm{def.}}{=} \sum_{\substack{l_1>\cdots>l_r>0\\ \mathrm{odd}}} a_{l_1\cdots l_r}(t)\cdot\det\begin{bmatrix} p_1^{l_1} & \cdots & p_1^{l_r}\\ \vdots & \ddots & \vdots\\ p_r^{l_1} & \cdots & p_r^{l_r}\end{bmatrix}.$$

$r=0$ のとき $a_{l_1\cdots l_r}(t)=1$ とする。

<u>主定理</u> $u$ を KdV 方程式 $u_{x_1}=(u_{x_0x_0}+3u^2)_{x_0}$ の解，$v$ を MKdV 方程式 $v_{x_1}=(v_{x_0x_0}-2v^3)_{x_0}$ の解とする(下の添字は微分を表わす)。$u=(2\log f)_{x_0x_0}$，$v=(\log\frac{f}{g})_{x_0}$ となる $f,g$ を適当に選べば

(6) $$\frac{f(x+t)f(x-t)}{f(x)^2}=\sum_{r=0}^{\infty}\sum_{\substack{l_1>\cdots>l_r>0\\ \mathrm{even}}} a_{l_1\cdots l_r}(t)\cdot K_{l_1\cdots l_r}[u],$$

(7) $$\frac{f(x+t)g(x-t)}{f(x)g(x)}=\sum_{r=0}^{\infty}\sum_{\substack{l_1>\cdots>l_r>0\\ \mathrm{odd}}} a_{l_1\cdots l_r}(t)\cdot K_{l_1\cdots l_r}[v]$$

が成り立つ。ここに $K_{l_1\cdots l_r}[u]$ は $u, u_{x_0}, u_{x_0x_0},\cdots$ を各々 weight $2,3,4,\cdots$ と数えてそれらの weight $l_1+\cdots+l_r$ の有重多項式，又 $K_{l_1\cdots l_r}[v]$ は $v, v_{x_0}, v_{x_0x_0},\cdots$ を各々 weight $1,2,3,\cdots$ と数えてそれらの weight $l_1+\cdots+l_r$ の有重多項式である(表2)。特に $r=0$ のとき $K_{l_1\cdots l_r}[u]=K_{l_1\cdots l_r}[v]=1$ とする。

$k=(k_0,k_1,k_2,\cdots)$ とし，$|k| \underset{\mathrm{def}}{=} k_0+3k_1+5k_2+\cdots$，$D^k=D_0^{k_0}D_1^{k_1}D_2^{k_2}\cdots$，$D_m=\frac{\partial}{\partial x_m}$ とする。$a_{l_1\cdots l_r}(t)\overset{\mathrm{def.}}{=}\sum_{|k|=l_1+\cdots+l_r} a_{l_1\cdots l_r:k}\frac{t^k}{k!}$ と書くとき主定理は次のように言いかえられる(左辺は広田氏[3]

の意味での微分とする)。

$$(6)_k \qquad \frac{D^k f\cdot f}{f^2} = \sum_{r=0}^{\infty} \sum_{\substack{l_1 > \cdots > l_r > 0 \\ \text{even} \\ l_1+\cdots+l_r=|k|}} a_{l_1\cdots l_r:k}\cdot K_{l_1\cdots l_r}[u] \;,$$

$$(7)_k \qquad \frac{D^k f\cdot g}{fg} = \sum_{r=0}^{\infty} \sum_{\substack{l_1 > \cdots > l_r > 0 \\ \text{odd} \\ l_1+\cdots+l_r=|k|}} a_{l_1\cdots l_r:k}\cdot K_{l_1\cdots l_r}[v] \;.$$

系　$P(D_x)=P(D_0, D_1, D_2, \cdots)$ が $D_0, D_1, D_2, \cdots$ を各々 weight $1, 3, 5, \cdots$ と数えてそれらの weight $n$ の斉重多項式であるとき($\Leftrightarrow$ は必要十分を表わす),

(8)
$$[\mathrm{KdV}] \quad P(D_x)f\cdot f=0$$
$$\Leftrightarrow P\left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_1}, \cdots\right) a_{l_1\cdots l_r}(t)=0 \quad \text{for } {}^\forall(l_1,\cdots,l_r) \text{ s.t. } l_1+\cdots+l_r=n,\ \underset{\text{even}}{l_1>\cdots>l_r>0}\,.$$

(9)
$$[\mathrm{MKdV}] \quad P(D_x)f\cdot g=0$$
$$\Leftrightarrow P\left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_1}, \cdots\right) a_{l_1\cdots l_r}(t)=0 \quad \text{for } {}^\forall(l_1,\cdots,l_r) \text{ s.t. } l_1+\cdots+l_r=n,\ \underset{\text{odd}}{l_1>\cdots>l_r>0}\,.$$

[NLS]

$t=(t_1, t_2, t_3, \cdots)$ において $t_1, t_2, t_3$ を各々 weight $1, 2, 3, \cdots$ と定める。

$x=(x_1, x_2, x_3, \cdots)$ についても同様とする。

(10) $\Delta(p_1,\cdots,p_{2r+1})\cdot\Phi(t;p_1,\cdots,p_{2r+1})$

$$\underset{\text{def.}}{=} \frac{1}{r!(r+1)!}\sum^{(2r+1)!} \pm\Delta(p_1,\cdots,p_r)\cdot\Delta(p_{r+1},\cdots,p_{2r+1})\,e^{\frac{t_1}{2}(-p_1-\cdots-p_r+p_{r+1}+\cdots+p_{2r+1})+\frac{t_2}{2}(-p_1^2-\cdots-p_r^2+p_{r+1}^2+\cdots+p_{2r+1}^2)+\cdots}\,.$$

(11) $\Delta(p_1,\cdots,p_{2r})\cdot\Phi(t;p_1,\cdots,p_{2r})$

$$\underset{\text{def.}}{=} \frac{1}{r!\,r!}\sum^{(2r)!} \pm\Delta(p_1,\cdots,p_r)\cdot\Delta(p_{r+1},\cdots,p_{2r})\,e^{\frac{t_1}{2}(-p_1-\cdots-p_r+p_{r+1}+\cdots+p_{2r})+\frac{t_2}{2}(-p_1^2-\cdots-p_r^2+p_{r+1}^2+\cdots+p_{2r}^2)+\cdots}\,.$$

$$(12)\quad \Delta(p_1,\cdots,p_{2r+1})\cdot\left(\Phi(t;p_1,\cdots,p_{2r+1})\right)^2 \overset{\text{def}}{=} \sum_{l_1>\cdots>l_{2r+1}\geq 0} a_{l_1\cdots l_{2r+1}}(t)\cdot\det\begin{bmatrix} p_1^{l_1} & \cdots & p_1^{l_{2r+1}} \\ \vdots & & \vdots \\ p_{2r+1}^{l_1} & \cdots & p_{2r+1}^{l_{2r+1}} \end{bmatrix}.$$

$$(13)\quad \Delta(p_1,\cdots,p_{2r})\cdot\left(\Phi(t;p_1,\cdots,p_{2r})\right)^2 \overset{\text{def}}{=} \sum_{l_1>\cdots>l_{2r}\geq 0} a_{l_1\cdots l_{2r}}(t)\cdot\det\begin{bmatrix} p_1^{l_1} & \cdots & p_1^{l_{2r}} \\ \vdots & & \vdots \\ p_{2r}^{l_1} & \cdots & p_{2r}^{l_{2r}} \end{bmatrix}.$$

$r=0$ のとき $a_{l_1\cdots l_{2r}}(t)=1$ とする。

$k=(k_1,k_2,k_3,\cdots)$ とし、$|k| \underset{\text{def}}{=} k_1+2k_2+3k_3+\cdots$, $D^k=D_1^{k_1}D_2^{k_2}D_3^{k_3}\cdots$, $D_m=\frac{\partial}{\partial x_m}$ とする。$a_{l_1\cdots l_{2r+1}}(t) \overset{\text{def}}{=} \sum_{|k|=l_1+\cdots+l_{2r+1}+r} a_{l_1\cdots l_{2r+1}:k}\dfrac{t^k}{k!}$, $a_{l_1\cdots l_{2r}}(t) \overset{\text{def}}{=} \sum_{|k|=l_1+\cdots+l_{2r}+r} a_{l_1\cdots l_{2r}:k}\dfrac{t^k}{k!}$ と書く。

<u>主定理</u> $v$ と $w$ (resp. $u$ と $v$) を NLS 方程式 $v_{x_2}=(v^2+w)_{x_1}$, $w_{x_2}=(v_{x_1x_1}+2vw)_{x_1}$ (resp. $u_{x_2}=(u_{x_1}+2uv)_{x_1}$, $v_{x_2}=(-v_{x_1}+v^2+u)_{x_1}$) の解とする。$u=(2\log f)_{x_1x_1}$, $v=(\log\frac{f}{g})_{x_1}$, $w=(\log fg)_{x_1x_1}$ となる $f,g$ を適当に選べば(このとき $u=v_{x_1}+w$ である)、

$$(14)_k\quad \frac{D^k f\cdot g}{fg} = \sum_{r=0}^{\infty}\sum_{\substack{l_1>\cdots>l_{2r+1}\geq 0\\ l_1+\cdots+l_{2r+1}=|k|-r}} a_{l_1\cdots l_{2r+1}:k}\cdot K_{l_1\cdots l_{2r+1}}[v,w],$$

$$(15)_k\quad \frac{D^k f\cdot f}{f^2} = \sum_{r=0}^{\infty}\sum_{\substack{l_1>\cdots>l_{2r}\geq 0\\ l_1+\cdots+l_{2r}=|k|-r}} a_{l_1\cdots l_{2r}:k}\cdot K_{l_1\cdots l_{2r}}[u,v]$$

が成り立つ。ここに、$K_{l_1\cdots l_{2r+1}}[v,w]$ は、$v, v_{x_1}, v_{x_1x_1},\cdots$ を weight $1,2,3,\cdots$, $w, w_{x_1}, w_{x_1x_1},\cdots$ を weight $2,3,4,\cdots$ と数えて、それらの weight $l_1+\cdots+l_{2r+1}+r$ の斉重多項式、$K_{l_1\cdots l_{2r}}[u,v]$ は、$u,u_{x_1},u_{x_1x_1},\cdots$, $v,v_{x_1},v_{x_1x_1},\cdots$ を weight $2,3,4,\cdots$, $1,2,3,\cdots$ と数えて、それらの weight $l_1+\cdots+l_{2r}+r$ の斉重多項式である(表4)。特に、$r=0$ のとき $K_{l_1\cdots l_{2r}}[u,v]=1$

とする。

<u>系</u>　$P(D_x)=P(D_1,D_2,D_3,\cdots)$ が $D_1,D_2,D_3\cdots$ を各々 weight $1,2,3,\cdots$ と数えて，それらの weight $n$ の斉重多項式であるとき，

$$\text{(16)}\quad \begin{aligned}&P(D_x)\,f\cdot f=0\\ \Leftrightarrow\ &P\left(\tfrac{\partial}{\partial t_1},\tfrac{\partial}{\partial t_2},\cdots\right)a_{\ell_1\cdots\ell_{2r+1}}(t)=0\quad \text{for } {}^{\forall}(\ell_1,\cdots,\ell_{2r+1})\ \text{s.t.}\ \ell_1+\cdots+\ell_{2r+1}+r=n,\ \ell_1>\cdots>\ell_{2r+1}\geqq 0.\end{aligned}$$

$$\text{(17)}\quad \begin{aligned}&P(D_x)\,f\cdot f=0\\ \Leftrightarrow\ &P\left(\tfrac{\partial}{\partial t_1},\tfrac{\partial}{\partial t_2},\cdots\right)a_{\ell_1\cdots\ell_{2r}}(t)=0\quad \text{for } {}^{\forall}(\ell_1,\cdots,\ell_{2r})\ \text{s.t.}\ \ell_1+\cdots+\ell_{2r}+r=n,\ \ell_1>\cdots>\ell_{2r}\geqq 0.\end{aligned}$$

$p(n)$ ＝ $n$ を自然数の和に分ける分け方の数，

$p^{\text{even}}_{\text{odd}}(n)$ ＝ $n$ を $\begin{matrix}\text{偶数}\\ \text{奇数}\end{matrix}$ の和に分ける分け方の数，

$q(n)$ ＝ $n$ を相異なる自然数の和に分ける分け方の数，

$q^{\text{even}}_{\text{odd}}(n)$ ＝ $n$ を相異なる $\begin{matrix}\text{偶数}\\ \text{奇数}\end{matrix}$ の和に分ける分け方の数

とおく。これらの母函数は次で与えられる。

$$\text{(18)}\quad \sum_{n=0}^{\infty}p(n)x^n=\frac{1}{(1-x)(1-x^2)(1-x^3)\cdots},$$

$$\text{(19)}^{\text{even}}\quad \sum_{n=0}^{\infty}p^{\text{even}}(n)x^n=\frac{1}{(1-x^2)(1-x^4)(1-x^6)\cdots},$$

$$\text{(19)}^{\text{odd}}\quad \sum_{n=0}^{\infty}p^{\text{odd}}(n)x^n=\frac{1}{(1-x)(1-x^3)(1-x^5)\cdots},$$

$$\text{(20)}\quad \sum_{n=0}^{\infty}q(n)x^n=(1+x)(1+x^2)(1+x^3)\cdots,$$

$$\text{(21)}^{\text{even}}\quad \sum_{n=0}^{\infty}q^{\text{even}}(n)x^n=(1+x^2)(1+x^4)(1+x^6)\cdots,$$

$$\text{(21)}^{\text{odd}}\quad \sum_{n=0}^{\infty}q^{\text{odd}}(n)x^n=(1+x)(1+x^3)(1+x^5)\cdots.$$

このとき，

(22)
$$p^{even}(n) = \begin{cases} p(\frac{n}{2}) & (n:\text{even}) \\ 0 & (n:\text{odd}) \end{cases}, \quad p^{odd}(n) = q(n),$$
$$q^{even}(n) = \begin{cases} q(\frac{n}{2}) & (n:\text{even}) \\ 0 & (n:\text{odd}) \end{cases}$$

の関係がある。又、これらの量の漸近的評価は、$C \underset{def}{=} \pi\sqrt{\frac{2}{3}}$ とするとき

(23)
$$\log p(n) \sim C\sqrt{n}, \quad \log q(n) \sim \log p^{even}_{odd}(n) \sim C\sqrt{\frac{n}{2}},$$
$$\log q^{even}_{odd}(n) \sim C\sqrt{\frac{n}{4}} \qquad (n \to \infty)$$

で与えられる。

今、

[KdV, MKdV] $\dim\{P(D_x) \mid P(D_x)$ は $D_0, D_1, D_2, \cdots$ の weight $n$ の斉重多項式$\} = q(n)$,

[NLS] $\dim\{P(D_x) \mid P(D_x)$ は $D_1, D_2, D_3, \cdots$ の weight $n$ の斉重多項式$\} = p(n)$

である。

定理(次元公式)

(24) [KdV] $\dim\left\{\frac{D^k f\cdot f}{f^2} \;\middle|\; k=(k_0,k_1,k_2,\cdots),\ |k|=n\right\} = q^{even}(n)$.

(25) [MKdV] $\dim\left\{\frac{D^k f\cdot g}{fg} \;\middle|\; k=(k_0,k_1,k_2,\cdots),\ |k|=n\right\} = q^{odd}(n)$.

(26) [NLS] $\dim\left\{\frac{D^k f\cdot g}{fg} \;\middle|\; k=(k_1,k_2,k_3,\cdots),\ |k|=n\right\} = q^{odd}(2n+1)$.

(27) $\dim\left\{\frac{D^k f\cdot f}{f^2} \;\middle|\; k=(k_1,k_2,k_3,\cdots),\ |k|=n\right\} = q^{odd}(2n)$.

(表8)に、0から59までの$n$について、これら$p(n)$, $p^{even}_{odd}(n)$, $q(n)$, $q^{even}_{odd}(n)$の数値をあげる[1]。

安定性定理　KdV方程式に対しては $m \geqq \frac{|k|-2}{4}$ 又は

$l_1 + 2m \geqq |k| + 2$ のとき，MKdV 方程式に対しては $m \geqq \frac{|k|-1}{4}$

又は $l_1 + 2m \geqq |k| + 3$ のとき $l_1 + \cdots + l_r = k_0 + 3k_1 + \cdots + (2m+1)(k_m+1)$，

$\underset{\substack{\text{even (KdV)} \\ \text{odd (MKdV)}}}{l_1 > \cdots > l_r > 0}$ を満たす $l$ と $k$ に対して，

$$a_{l_1 \cdots l_r : k_0, \cdots k_{m-1}, k_m+1} = a_{l_1+2, \cdots, l_r : k_0, \cdots, k_{m-1}, k_m, 1} = a_{l_1+4, \cdots, l_r : k_0, \cdots, k_{m-1}, k_m, 0, 1}$$

$$= \cdots = a_{l_1+2\nu, \cdots, l_r : k_0, \cdots, k_{m-1}, k_m, \underbrace{0, \cdots, 0}_{\nu-1}, 1}$$

が成り立つ。

以下に，$\frac{D^k f \cdot f}{f^2}$，$\frac{D^k f \cdot g}{fg}$ 等を，各々の場合に，basis $\{K_{l_1 \cdots l_r}\}$ の一次結合で表わした表を挙げる（表 1, 3）。Bilinear Equations はこれらの表に含まれているが，便利のため，一部を書き出しておく（表 5, 6, 7）。

(表1)

| | MKdV | KdV |
|---|---|---|
| | 1 | 1 |
| 1 | 1 | 1 |

| | $K_1$ |
|---|---|
| $D_0$ | 1 |

| | 0 | $K_2$ |
|---|---|---|
| $D_0^2$ | 0 | 1 |

| | $K_3$ |
|---|---|
| $D_0^3$ | 1 |
| $D_1$ | 1 |

| | $K_{3,1}$ | $K_4$ |
|---|---|---|
| $D_0^4$ | 2 | 1 |
| $D_0 D_1$ | -1 | 1 |

| | $K_5$ |
|---|---|
| $D_0^5$ | 1 |
| $D_0^2 D_1$ | 1 |
| $D_2$ | 1 |

| | $K_{5,1}$ | $K_6$ | $K_{4,2}$ |
|---|---|---|---|
| $D_0^6$ | 4 | 1 | 5 |
| $D_0^3 D_1$ | 1 | 1 | -1 |
| $D_1^2$ | -2 | 1 | 2 |
| $D_0 D_2$ | -1 | 1 | 0 |

| | MKdV | KdV |
|---|---|---|
| | $K_7$ | |
| $D_0^7$ | 1 | |
| $D_0^4 D_1$ | 1 | |
| $D_0 D_1^2$ | 1 | |
| $D_0^2 D_2$ | 1 | |
| $D_3$ | 1 | |

| | $K_{7,1}$ | $K_{5,3}$ | $K_8$ | $K_{6,2}$ |
|---|---|---|---|---|
| $D_0^8$ | 6 | 14 | 1 | 14 |
| $D_0^5 D_1$ | 3 | -1 | 1 | 2 |
| $D_0^2 D_1^2$ | 0 | 2 | 1 | -1 |
| $D_0^3 D_2$ | 1 | -1 | 1 | -1 |
| $D_1 D_2$ | -2 | -1 | 1 | [illegible] |
| $D_0 D_3$ | -1 | 0 | 1 | 0 |

| | $K_9$ | $K_{5,3,1}$ |
|---|---|---|
| $D_0^9$ | 1 | 42 |
| $D_0^6 D_1$ | 1 | -6 |
| $D_0^3 D_1^2$ | 1 | 0 |
| $D_1^3$ | 1 | 6 |
| $D_0^4 D_2$ | 1 | 2 |
| $D_0 D_1 D_2$ | 1 | -1 |
| $D_0^2 D_3$ | 1 | 0 |
| $D_4$ | 1 | 0 |

(表1)

| | MKdV | | KdV | | |
|---|---|---|---|---|---|
| | $K_{9,1}$ | $K_{7,3}$ | $K_{10}$ | $K_{8,2}$ | $K_{6,4}$ |
| $D_0^{10}$ | 8 | 48 | 1 | 27 | 42 |
| $D_0^7 D_1$ | 5 | 6 | 1 | 9 | 0 |
| $D_0^4 D_1^2$ | 2 | 0 | 1 | 0 | 3 |
| $D_0 D_1^3$ | −1 | 3 | 1 | 0 | −3 |
| $D_0^5 D_2$ | 3 | −2 | 1 | 2 | −3 |
| $D_0^2 D_1 D_2$ | 0 | 1 | 1 | −1 | 0 |
| $D_2^2$ | −2 | −2 | 1 | 2 | 2 |
| $D_0^3 D_3$ | 1 | −1 | 1 | −1 | 0 |
| $D_1 D_3$ | −2 | −1 | 1 | 2 | 0 |
| $D_0 D_4$ | −1 | 0 | 1 | 0 | 0 |

| | $K_{11}$ | $K_{7,3,1}$ |
|---|---|---|
| $D_0^{11}$ | 1 | 198 |
| $D_0^8 D_1$ | 1 | 6 |
| $D_0^5 D_1^2$ | 1 | −6 |
| $D_0^2 D_1^3$ | 1 | 0 |
| $D_0^6 D_2$ | 1 | −2 |
| $D_0^3 D_1 D_2$ | 1 | 1 |
| $D_1^2 D_2$ | 1 | 4 |
| $D_0 D_2^2$ | 1 | −2 |
| $D_0^4 D_3$ | 1 | 2 |
| $D_0 D_1 D_3$ | 1 | −1 |
| $D_0^2 D_4$ | 1 | 0 |
| $D_5$ | 1 | 0 |

| | MKdV | | | KdV | | | |
|---|---|---|---|---|---|---|---|
| | $K_{11,1}$ | $K_{9,3}$ | $K_{7,5}$ | $K_{12}$ | $K_{10,2}$ | $K_{8,4}$ | $K_{6,4,2}$ |
| $D_0^{12}$ | 10 | 110 | 132 | 1 | 44 | 165 | 462 |
| $D_0^9 D_1$ | 7 | 24 | 6 | 1 | 20 | 21 | −42 |
| $D_0^6 D_1^2$ | 4 | 2 | 6 | 1 | 5 | 3 | 3 |
| $D_0^3 D_1^3$ | 1 | 2 | −3 | 1 | −1 | 3 | 3 |
| $D_1^4$ | −2 | 2 | 6 | 1 | 2 | 6 | 12 |
| $D_0^7 D_2$ | 5 | 5 | −8 | 1 | 9 | −5 | 7 |
| $D_0^4 D_1 D_2$ | 2 | −1 | 1 | 1 | 0 | 1 | −2 |
| $D_0 D_1^2 D_2$ | −1 | 2 | 1 | 1 | 0 | −2 | −2 |
| $D_0^2 D_2^2$ | 0 | 0 | 2 | 1 | −1 | 0 | 2 |
| $D_0^5 D_3$ | 3 | −2 | −1 | 1 | 2 | −3 | 0 |
| $D_0^2 D_1 D_3$ | 0 | 1 | −1 | 1 | −1 | 0 | 0 |
| $D_2 D_3$ | −2 | −2 | −1 | 1 | 2 | 2 | 0 |
| $D_0^3 D_4$ | 1 | −1 | 0 | 1 | −1 | 0 | 0 |
| $D_1 D_4$ | −2 | −1 | 0 | 1 | 2 | 0 | 0 |
| $D_0 D_5$ | −1 | 0 | 0 | 1 | 0 | 0 | 0 |

(表1)

| | MKdV | | |
|---|---|---|---|
| | $K_{13}$ | $K_{9,3,1}$ | $K_{7,5,1}$ |
| $D_0^{13}$ | 1 | 572 | 858 |
| $D_0^{10}D_1$ | 1 | 92 | −6 |
| $D_0^7D_1^2$ | 1 | −10 | 12 |
| $D_0^4D_1^3$ | 1 | −4 | −6 |
| $D_0D_1^4$ | 1 | 2 | −6 |
| $D_0^8D_2$ | 1 | 12 | −22 |
| $D_0^5D_1D_2$ | 1 | −3 | −1 |
| $D_0^2D_1^2D_2$ | 1 | 0 | 2 |
| $D_0^3D_2^2$ | 1 | 2 | −2 |
| $D_1D_2^2$ | 1 | 2 | 4 |
| $D_0^6D_3$ | 1 | −2 | 4 |
| $D_0^3D_1D_3$ | 1 | 1 | 1 |
| $D_1^2D_3$ | 1 | 4 | −2 |
| $D_0D_2D_3$ | 1 | −2 | −1 |
| $D_0^4D_4$ | 1 | 2 | 0 |
| $D_0D_1D_4$ | 1 | −1 | 0 |
| $D_0^2D_5$ | 1 | 0 | 0 |
| $D_6$ | 1 | 0 | 0 |

| | MKdV | | | KdV | | | | |
|---|---|---|---|---|---|---|---|---|
| | $K_{13,1}$ | $K_{11,3}$ | $K_{9,5}$ | $K_{14}$ | $K_{12,2}$ | $K_{10,4}$ | $K_{8,6}$ | $K_{8,4,2}$ |
| $D_0^{14}$ | 12 | 208 | 572 | 1 | 65 | 429 | 429 | 3003 |
| $D_0^{11}D_1$ | 9 | 76 | 77 | 1 | 35 | 99 | 33 | 33 |
| $D_0^8D_1^2$ | 6 | 16 | 14 | 1 | 14 | 12 | 15 | −30 |
| $D_0^5D_1^3$ | 3 | 1 | 5 | 1 | 2 | 6 | −3 | 6 |
| $D_0^2D_1^4$ | 0 | 4 | −4 | 1 | −1 | 0 | 6 | −6 |
| $D_0^9D_2$ | 7 | 28 | −13 | 1 | 20 | 14 | −21 | −12 |
| $D_0^6D_1D_2$ | 4 | 1 | 2 | 1 | 5 | −1 | 3 | 3 |
| $D_0^3D_1^2D_2$ | 1 | 1 | −1 | 1 | −1 | 2 | 0 | 0 |
| $D_1^3D_2$ | −2 | 1 | $\bar{5}$ | 1 | 2 | −4 | −3 | 6 |
| $D_0^4D_2^2$ | 2 | −2 | 2 | 1 | 0 | −1 | 4 | −2 |
| $D_0D_1D_2^2$ | −1 | 1 | 2 | 1 | 0 | −1 | −2 | −2 |
| $D_0^7D_3$ | 5 | 5 | −9 | 1 | 9 | −5 | −5 | 7 |
| $D_0^4D_1D_3$ | 2 | −1 | 0 | 1 | 0 | 1 | −2 | −2 |
| $D_0D_1^2D_3$ | −1 | 2 | 0 | 1 | 0 | −2 | 1 | −2 |
| $D_0^2D_2D_3$ | 0 | 0 | 1 | 1 | −1 | 0 | 0 | 2 |
| $D_3^2$ | −2 | −2 | −2 | 1 | 2 | 2 | 2 | 0 |
| $D_0^5D_4$ | 3 | −2 | −1 | 1 | 2 | −3 | 0 | 0 |
| $D_0^2D_1D_4$ | 0 | 1 | −1 | 1 | −1 | 0 | 0 | 0 |
| $D_2D_4$ | −2 | −2 | −1 | 1 | 2 | 2 | 0 | 0 |
| $D_0^3D_5$ | 1 | −1 | 0 | 1 | −1 | 0 | 0 | 0 |
| $D_1D_5$ | −2 | −1 | 0 | 1 | 2 | 0 | 0 | 0 |
| $D_0D_6$ | −1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |

(表1)

KdV

| | $K_{16}$ | $K_{14,2}$ | $K_{12,4}$ | $K_{10,6}$ | $K_{10,4,2}$ | $K_{8,6,2}$ |
|---|---|---|---|---|---|---|
| $D_0^{16}$ | 1 | 90 | 910 | 2002 | 11440 | 15444 |
| $D_0^{13}D_1$ | 1 | 54 | 286 | 286 | 1144 | 0 |
| $D_0^{10}D_1^2$ | 1 | 27 | 58 | 55 | −98 | 81 |
| $D_0^7D_1^3$ | 1 | 9 | 10 | 13 | −8 | −27 |
| $D_0^4D_1^4$ | 1 | 0 | 7 | −2 | 10 | 0 |
| $D_0D_1^5$ | 1 | 0 | −5 | 10 | 10 | 0 |
| $D_0^{11}D_2$ | 1 | 35 | 90 | −33 | 110 | −231 |
| $D_0^8D_1D_2$ | 1 | 14 | 6 | 6 | −16 | 0 |
| $D_0^5D_1^2D_2$ | 1 | 2 | 3 | 0 | 2 | 6 |
| $D_0^2D_1^3D_2$ | 1 | −1 | 0 | 3 | 2 | 3 |
| $D_0^6D_2^2$ | 1 | 5 | −5 | 7 | 5 | −6 |
| $D_0^3D_1D_2^2$ | 1 | −1 | 1 | 1 | −1 | 0 |
| $D_1^2D_2^2$ | 1 | 2 | −2 | −5 | 2 | 6 |
| $D_0D_2^3$ | 1 | 0 | 0 | −3 | 0 | −6 |
| $D_0^9D_3$ | 1 | 20 | 14 | −28 | −12 | 30 |
| $D_0^6D_1D_3$ | 1 | 5 | −1 | −1 | 3 | 0 |
| $D_0^3D_1^2D_3$ | 1 | −1 | 2 | −1 | 0 | −3 |
| $D_1^3D_3$ | 1 | 2 | −4 | −1 | 6 | −6 |
| $D_0^4D_2D_3$ | 1 | 0 | −1 | 2 | −2 | 0 |

| | $K_{16}$ | $K_{14,2}$ | $K_{12,4}$ | $K_{10,6}$ | $K_{10,4,2}$ | $K_{8,6,2}$ |
|---|---|---|---|---|---|---|
| $D_0D_1D_2D_3$ | 1 | 0 | −1 | −1 | −2 | 0 |
| $D_0^2D_3^2$ | 1 | −1 | 0 | 0 | 2 | 2 |
| $D_0^7D_4$ | 1 | 9 | −5 | −5 | 7 | 0 |
| $D_0^4D_1D_4$ | 1 | 0 | 1 | −2 | −2 | 0 |
| $D_0D_1^2D_4$ | 1 | 0 | −2 | 1 | −2 | 0 |
| $D_0^2D_2D_4$ | 1 | −1 | 0 | 0 | 2 | 0 |
| $D_3D_4$ | 1 | 2 | 2 | 2 | 0 | 0 |
| $D_0^5D_5$ | 1 | 2 | −3 | 0 | 0 | 0 |
| $D_0^2D_1D_5$ | 1 | −1 | 0 | 0 | 0 | 0 |
| $D_2D_5$ | 1 | 2 | 2 | 0 | 0 | 0 |
| $D_0^3D_6$ | 1 | −1 | 0 | 0 | 0 | 0 |
| $D_1D_6$ | 1 | 2 | 0 | 0 | 0 | 0 |
| $D_0D_7$ | 1 | 0 | 0 | 0 | 0 | 0 |

(表2) KdV

| | $u$ |
|---|---|
| $K_2$ | 1 |

| | $u_2$ | $u^2$ |
|---|---|---|
| $K_4$ | 1 | 3 |

| | $u_4$ | $uu_2$ | $u_1^2$ | $u^3$ |
|---|---|---|---|---|
| $K_6$ | 1 | 10 | 5 | 10 |
| $K_{4,2}$ | 0 | 1 | −1 | 1 |

| | $u_6$ | $uu_4$ | $u_1u_3$ | $u_2^2$ | $u^2u_2$ | $uu_1^2$ | $u^4$ |
|---|---|---|---|---|---|---|---|
| $K_8$ | 1 | 14 | 28 | 21 | 70 | 70 | 35 |
| $K_{6,2}$ | 0 | 1 | −2 | 1 | 10 | −5 | 5 |

| | $u_8$ | $uu_6$ | $u_1u_5$ | $u_2u_4$ | $u_3^2$ | $u^2u_4$ | $uu_1u_3$ | $uu_2^2$ | $u_1^2u_2$ | $u^3u_2$ | $u^2u_1^2$ | $u^5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_{10}$ | 1 | 18 | 54 | 114 | 69 | 126 | 504 | 378 | 462 | 420 | 630 | 126 |
| $K_{8,2}$ | 0 | 1 | −2 | 2 | −1 | 14 | 0 | 35 | −28 | 70 | 0 | 21 |
| $K_{6,4}$ | 0 | 0 | 0 | 1 | −1 | 3 | −12 | 6 | 7 | 20 | −15 | 6 |

| | $u_{10}$ | $uu_8$ | $u_1u_7$ | $u_2u_6$ | $u_3u_5$ | $u_4^2$ | $u^2u_6$ | $uu_1u_5$ | $uu_2u_4$ | $uu_3^2$ | $u_1^2u_4$ | $u_1u_2u_3$ | $u_2^3$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_{12}$ | 1 | 22 | 88 | 242 | 418 | 253 | 198 | 1188 | 2508 | 1518 | 1650 | 5676 | 1342 |
| $K_{10,2}$ | 0 | 1 | −2 | 2 | −2 | 1 | 18 | 18 | 150 | 51 | −72 | −120 | 40 |
| $K_{8,4}$ | 0 | 0 | 0 | 1 | −2 | 1 | 3 | −12 | 26 | −20 | 12 | −8 | 19 |
| $K_{6,4,2}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | −1 | 2 | −1 |

| | $u^3u_4$ | $u^2u_1u_3$ | $u^2u_2^2$ | $uu_1^2u_2$ | $u_1^4$ | $u^4u_2$ | $u^3u_1^2$ | $u^6$ |
|---|---|---|---|---|---|---|---|---|
| $K_{12}$ | 924 | 5544 | 4158 | 10164 | 1155 | 2310 | 4620 | 462 |
| $K_{10,2}$ | 126 | 252 | 504 | −42 | −105 | 420 | 210 | 84 |
| $K_{8,4}$ | 42 | −84 | 147 | −70 | 35 | 175 | −70 | 35 |
| $K_{6,4,2}$ | 1 | −6 | 3 | 7 | −5 | 5 | −5 | 1 |

(表2) MKdV

| | $v$ |
|---|---|
| $K_1$ | 1 |

| | $v_2$ | $v^3$ |
|---|---|---|
| $K_3$ | 1 | −2 |

| | $vv_2$ | $v_1^2$ | $v^4$ |
|---|---|---|---|
| $K_{3,1}$ | 1 | −1 | −1 |

| | $v_4$ | $v^2v_2$ | $vv_1^2$ | $v^5$ |
|---|---|---|---|---|
| $K_5$ | 1 | −10 | −10 | 6 |

| | $vv_4$ | $v_1v_3$ | $v_2^2$ | $v^3v_2$ | $v^6$ |
|---|---|---|---|---|---|
| $K_{5,1}$ | 1 | −2 | 1 | −10 | 4 |

| | $v_6$ | $v^2v_4$ | $vv_1v_3$ | $vv_2^2$ | $v_1^2v_2$ | $v^4v_2$ | $v^3v_1^2$ | $v^7$ |
|---|---|---|---|---|---|---|---|---|
| $K_7$ | 1 | −14 | −56 | −42 | −70 | 70 | 140 | −20 |

| | $vv_6$ | $v_1v_5$ | $v_2v_4$ | $v_3^2$ | $v^3v_4$ | $v^2v_1v_3$ | $v^2v_2^2$ | $vv_1^2v_2$ | $v_1^4$ | $v^5v_2$ | $v^4v_1^2$ | $v^8$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_{7,1}$ | 1 | −2 | 2 | −1 | −14 | −28 | −56 | −14 | 21 | 70 | 70 | −15 |
| $K_{5,3}$ | 0 | 0 | 1 | −1 | −2 | 12 | −6 | −14 | 1 | 14 | −10 | −3 |

| | $v_8$ | $v^2v_6$ | $vv_1v_5$ | $vv_2v_4$ | $vv_3^2$ | $v_1^2v_4$ | $v_1v_2v_3$ | $v_2^3$ |
|---|---|---|---|---|---|---|---|---|
| $K_9$ | 1 | −18 | −108 | −228 | −138 | −210 | −756 | −182 |
| $K_{5,3,1}$ | 0 | 0 | 0 | 1 | −1 | −1 | 2 | −1 |

| | $v^4v_4$ | $v^3v_1v_3$ | $v^3v_2^2$ | $v^2v_1^2v_2$ | $vv_1^4$ | $v^6v_2$ | $v^5v_1^2$ | $v^9$ |
|---|---|---|---|---|---|---|---|---|
| $K_9$ | 126 | 1008 | 756 | 3108 | 798 | −420 | −1260 | 70 |
| $K_{5,3,1}$ | −1 | 8 | −4 | −14 | 11 | 6 | −6 | −1 |

但し、$u_j$ は $\left(\frac{\partial}{\partial x_0}\right)^j u = u_{\underbrace{x_0 \cdots x_0}_{j}}$ の略記であり、$v_j$ も同様である。
Matsuno [5] の (3.2a) ~ (3.2d)、Matsuno [6] の Appendix A, B, C 参照。

(表3) NLS

| | f·g | f·f |
|---|---|---|
| | $K_0$ | $1$ |
| $1$ | 1 | 1 |

| | f·g | f·f |
|---|---|---|
| | $K_1$ | $0$ |
| $D_1$ | 1 | 0 |

| | f·g | f·f |
|---|---|---|
| | $K_2$ | $K_{1,0}$ |
| $D_1^2$ | 1 | 1 |
| $D_2$ | 1 | 0 |

| | f·g | f·f |
|---|---|---|
| | $K_3$ | $K_{2,0}$ |
| $D_1^3$ | 1 | 0 |
| $D_1D_2$ | 1 | 1 |
| $D_3$ | 1 | 0 |

| | f·g | | f·f | |
|---|---|---|---|---|
| | $K_4$ | $2K_{2,1,0}$ | $K_{3,0}$ | $K_{2,1}$ |
| $D_1^4$ | 1 | 3 | 1 | −3 |
| $D_1^2D_2$ | 1 | −1 | 0 | 0 |
| $D_2^2$ | 1 | 1 | 1 | 1 |
| $D_1D_3$ | 1 | 0 | 1 | 0 |
| $D_4$ | 1 | 0 | 0 | 0 |

| | f·g | | f·f | |
|---|---|---|---|---|
| | $K_5$ | $2K_{3,1,0}$ | $K_{4,0}$ | $K_{3,1}$ |
| $D_1^5$ | 1 | 5 | 0 | 0 |
| $D_1^3D_2$ | 1 | 1 | 1 | −2 |
| $D_1D_2^2$ | 1 | −1 | 0 | 0 |
| $D_1^2D_3$ | 1 | −1 | 0 | 0 |
| $D_2D_3$ | 1 | 1 | 1 | 1 |
| $D_1D_4$ | 1 | 0 | 1 | 0 |
| $D_5$ | 1 | 0 | 0 | 0 |

| | f·g | | | f·f | | |
|---|---|---|---|---|---|---|
| | $K_6$ | $2K_{4,1,0}$ | $2K_{3,2,0}$ | $K_{5,0}$ | $K_{4,1}$ | $K_{3,2}$ |
| $D_1^6$ | 1 | 10 | −5 | 1 | −5 | 10 |
| $D_1^4D_2$ | 1 | 2 | 3 | 0 | 0 | 0 |
| $D_1^2D_2^2$ | 1 | 0 | 1 | 1 | −1 | −2 |
| $D_2^3$ | 1 | 0 | −3 | 0 | 0 | 0 |
| $D_1^3D_3$ | 1 | 1 | −2 | 1 | −2 | 1 |
| $D_1D_2D_3$ | 1 | −1 | 0 | 0 | 0 | 0 |
| $D_3^2$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $D_1^2D_4$ | 1 | −1 | 0 | 0 | 0 | 0 |
| $D_2D_4$ | 1 | 1 | 0 | 1 | 1 | 0 |
| $D_1D_5$ | 1 | 0 | 0 | 1 | 0 | 0 |
| $D_6$ | 1 | 0 | 0 | 0 | 0 | 0 |

(表3)　　NLS

| | f•g | | | | f•f | | |
|---|---|---|---|---|---|---|---|
| | $K_7$ | $2K_{5,1,0}$ | $2K_{4,3,0}$ | $2K_{3,3,1}$ | $K_{6,0}$ | $K_{5,1}$ | $K_{4,2}$ |
| $D_1^7$ | 1 | 14 | 0 | -35 | 0 | 0 | 0 |
| $D_1^5D_2$ | 1 | 6 | 0 | 5 | 1 | -4 | 5 |
| $D_1^3D_2^2$ | 1 | 0 | 4 | -1 | 0 | 0 | 0 |
| $D_1D_2^3$ | 1 | 0 | 0 | 3 | 1 | 0 | -3 |
| $D_1^4D_3$ | 1 | 2 | 0 | 1 | 0 | 0 | 0 |
| $D_1^2D_2D_3$ | 1 | 0 | 0 | -1 | 1 | -1 | -1 |
| $D_2^2D_3$ | 1 | 0 | -2 | -1 | 0 | 0 | 0 |
| $D_1D_3^2$ | 1 | -1 | 0 | 1 | 0 | 0 | 0 |
| $D_1^3D_4$ | 1 | 1 | -2 | 0 | 1 | -2 | 1 |
| $D_1D_2D_4$ | 1 | -1 | 0 | 0 | 0 | 0 | 0 |
| $D_3D_4$ | 1 | 1 | 1 | 0 | 1 | 1 | 1 |
| $D_1^2D_5$ | 1 | -1 | 0 | 0 | 0 | 0 | 0 |
| $D_2D_5$ | 1 | 1 | 0 | 0 | 1 | 1 | 0 |
| $D_1D_6$ | 1 | 0 | 0 | 0 | 1 | 0 | 0 |
| $D_7$ | 1 | 0 | 0 | 0 | 0 | 0 | 0 |

f•g

| | $K_8$ | $2K_{6,1,0}$ | $2K_{5,2,0}$ | $2K_{4,3,0}$ | $2K_{4,2,1}$ |
|---|---|---|---|---|---|
| $D_1^8$ | 1 | 21 | -7 | 35 | -70 |
| $D_1^6D_2$ | 1 | 9 | 5 | -5 | -10 |
| $D_1^4D_2^2$ | 1 | 3 | 3 | -1 | 4 |
| $D_1^2D_2^3$ | 1 | -1 | 3 | 3 | 0 |
| $D_2^4$ | 1 | 1 | -3 | 3 | 6 |
| $D_1^5D_3$ | 1 | 6 | -4 | 5 | 5 |
| $D_1^3D_2D_3$ | 1 | 0 | 2 | 1 | -1 |
| $D_1D_2^2D_3$ | 1 | 0 | 0 | -1 | 1 |
| $D_1^2D_3^2$ | 1 | 0 | -1 | 2 | -1 |
| $D_2D_3^2$ | 1 | 0 | -1 | -2 | -1 |
| $D_1^4D_4$ | 1 | 2 | 0 | -3 | 1 |
| $D_1^2D_2D_4$ | 1 | 0 | 0 | -1 | -1 |
| $D_2^2D_4$ | 1 | 0 | -2 | 1 | -1 |
| $D_1D_3D_4$ | 1 | -1 | 0 | 0 | 1 |
| $D_4^2$ | 1 | 1 | 1 | 1 | 0 |
| $D_1^3D_5$ | 1 | 1 | -2 | 0 | 0 |
| $D_1D_2D_5$ | 1 | -1 | 0 | 0 | 0 |
| $D_3D_5$ | 1 | 1 | 1 | 0 | 0 |
| $D_1^2D_6$ | 1 | -1 | 0 | 0 | 0 |
| $D_2D_6$ | 1 | 1 | 0 | 0 | 0 |
| $D_1D_7$ | 1 | 0 | 0 | 0 | 0 |
| $D_8$ | 1 | 0 | 0 | 0 | 0 |

(表4)　　　　　　NLS $(f \cdot g)$

$\left(\frac{\partial}{\partial x_1}\right)^j v = v_{\underbrace{x_1 \cdots x_1}_{j}}$ を $v_j$ と略記する。$u_j$, $w_j$ も同様である。

| | $v$ |
|---|---|
| $K_1$ | 1 |

| | $w$ | $v^2$ |
|---|---|---|
| $K_2$ | 1 | 1 |

| | $v_2$ | $vw$ | $v^3$ |
|---|---|---|---|
| $K_3$ | 1 | 3 | 1 |

| | $w_2$ | $vv_2$ | $v_1^2$ | $w^2$ | $v^2w$ | $v^4$ |
|---|---|---|---|---|---|---|
| $K_4$ | 1 | 4 | $\frac{3}{2}$ | $\frac{3}{2}$ | 6 | 1 |
| $2K_{2,1,0}$ | 0 | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 |

| | $v_4$ | $vw_2$ | $v_1w_1$ | $v_2w$ | $v^2v_2$ | $vv_1^2$ | $vw^2$ | $v^3w$ | $v^5$ |
|---|---|---|---|---|---|---|---|---|---|
| $K_5$ | 1 | 5 | 5 | 5 | 10 | $\frac{15}{2}$ | $\frac{15}{2}$ | 10 | 1 |
| $2K_{3,1,0}$ | 0 | 0 | $-1$ | 1 | 0 | $-\frac{3}{2}$ | $\frac{3}{2}$ | 0 | 0 |

| | $w_4$ | $vv_4$ | $v_1v_3$ | $v_2^2$ | $ww_2$ | $w_1^2$ | $v^2w_2$ | $vv_1w_1$ | $vv_2w$ | $v_1^2w$ | $w^3$ | $v^3v_2$ | $v^2v_1^2$ | $v^2w^2$ | $v^4w$ | $v^6$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_6$ | 1 | 6 | 10 | $\frac{15}{2}$ | 5 | $\frac{5}{2}$ | 15 | 30 | 30 | $\frac{25}{2}$ | $\frac{5}{2}$ | 20 | $\frac{45}{2}$ | $\frac{45}{2}$ | 15 | 1 |
| $2K_{4,1,0}$ | 0 | 0 | $-1$ | $\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | 0 | $-4$ | 4 | $-1$ | 1 | 0 | $-3$ | 3 | 0 | 0 |
| $2K_{3,2,0}$ | 0 | 0 | 0 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | 0 | $-2$ | 2 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{3}{2}$ | $\frac{3}{2}$ | 0 | 0 |

| | $v_6$ | $vw_4$ | $v_1w_3$ | $v_2w_2$ | $v_3w_1$ | $v_4w$ | $v^2v_4$ | $vv_1v_3$ | $vv_2^2$ | $v_1^2v_2$ | $vww_2$ | $vw_1^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_7$ | 1 | 7 | 14 | 21 | 14 | 7 | 21 | 70 | $\frac{105}{2}$ | $\frac{105}{2}$ | 35 | $\frac{35}{2}$ |
| $2K_{5,1,0}$ | 0 | 0 | $-1$ | 1 | $-1$ | 1 | 0 | $-5$ | $\frac{5}{2}$ | $-5$ | 5 | $-\frac{5}{2}$ |
| $2K_{4,2,0}$ | 0 | 0 | 0 | 1 | $-1$ | 0 | 0 | $-2$ | 3 | 2 | 2 | $-3$ |
| $2K_{3,2,1}$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ |

| | $v_1ww_1$ | $v_2w^2$ | $v^3w_2$ | $v^2v_1w_1$ | $v^2v_2w$ | $vv_1^2w$ | $vw^3$ | $v^4v_2$ | $v^3v_1^2$ | $v^3w^2$ | $v^5w$ | $v^7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_7$ | 35 | $\frac{35}{2}$ | 35 | 105 | 105 | $\frac{175}{2}$ | $\frac{35}{2}$ | 35 | $\frac{105}{2}$ | $\frac{105}{2}$ | 21 | 1 |
| $2K_{5,1,0}$ | 0 | 5 | 0 | $-10$ | 10 | $-5$ | 5 | 0 | $-5$ | 5 | 0 | 0 |
| $2K_{4,2,0}$ | $-2$ | 0 | 0 | $-8$ | 8 | 0 | 0 | 0 | $-4$ | 4 | 0 | 0 |
| $2K_{3,2,1}$ | 1 | $-\frac{1}{2}$ | 0 | $-1$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 |

(表4) NLS (f·f)

| | $u$ |
|---|---|
| $K_{1,0}$ | 1 |

| | $u_1$ | $uv$ |
|---|---|---|
| $K_{2,0}$ | 1 | 2 |

| | $u_2$ | $u_1v$ | $u^2$ | $uv^2$ |
|---|---|---|---|---|
| $K_{3,0}$ | 1 | 3 | $\frac{3}{2}$ | 3 |
| $K_{2,1}$ | 0 | 1 | $-\frac{1}{2}$ | 1 |

| | $u_3$ | $u_2v$ | $u_1v_1$ | $uv_2$ | $uu_1$ | $u_1v^2$ | $u^2v$ | $uv^3$ |
|---|---|---|---|---|---|---|---|---|
| $K_{4,0}$ | 1 | 4 | 2 | 2 | 3 | 6 | 6 | 4 |
| $K_{3,1}$ | 0 | 1 | -1 | 0 | 0 | 3 | 0 | 2 |

| | $u_4$ | $u_3v$ | $u_2v_1$ | $u_1v_2$ | $uu_2$ | $u_1^2$ | $u_2v^2$ | $u_1vv_1$ | $uu_1v$ | $uv_1^2$ | $uvv_2$ | $u^3$ | $u_1v^3$ | $u^2v^2$ | $uv^4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_{5,0}$ | 1 | 5 | 5 | 5 | 5 | $\frac{5}{2}$ | 10 | 10 | 15 | 5 | 10 | $\frac{5}{2}$ | 10 | 15 | 5 |
| $K_{4,1}$ | 0 | 1 | -1 | 1 | 0 | $-\frac{1}{2}$ | 4 | -2 | 3 | -1 | 2 | $-\frac{1}{2}$ | 6 | 3 | 3 |
| $K_{3,2}$ | 0 | 0 | -1 | 0 | 1 | $-\frac{1}{2}$ | 1 | -2 | 0 | -1 | 0 | 1 | 2 | 0 | 1 |

| | $u_5$ | $u_4v$ | $u_3v_1$ | $u_2v_2$ | $u_1v_3$ | $uv_4$ | $uu_3$ | $u_1u_2$ | $u_3v^2$ | $u_2vv_1$ | $u_1vv_2$ | $u_1v_1^2$ | $uu_2v$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_{6,0}$ | 1 | 6 | 9 | 11 | 4 | 2 | 5 | 10 | 15 | 30 | 30 | 15 | 30 |
| $K_{5,1}$ | 0 | 1 | -1 | 1 | -1 | 0 | 0 | 0 | 5 | 0 | 10 | 0 | 5 |
| $K_{4,2}$ | 0 | 0 | -1 | 1 | 0 | 0 | 1 | -1 | 1 | -6 | 2 | -3 | 4 |

| | $u_1^2v$ | $uu_1v_1$ | $u^2v_2$ | $u^2u_1$ | $u_2v^3$ | $u_1v^2v_1$ | $uv^2v_2$ | $uvv_1^2$ | $uu_1v^2$ | $u^3v$ | $u_1v^4$ | $u^2v^3$ | $uv^5$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $K_{6,0}$ | 15 | 20 | 10 | $\frac{15}{2}$ | 20 | 30 | 30 | 30 | 45 | 15 | 15 | 30 | 6 |
| $K_{5,1}$ | 0 | -20 | 0 | 0 | 10 | 0 | 10 | 0 | 15 | 0 | 10 | 10 | 4 |
| $K_{4,2}$ | -3 | 15 | 2 | $\frac{3}{2}$ | 4 | -6 | 2 | -6 | 3 | 3 | 5 | 2 | 2 |

(表5)　　　KdV 方程式

$D_0(D_0^3-D_1)f\cdot f=0$, $(D_0^3-D_1)(D_0^3+2D_1)f\cdot f=0$, $D_0^2(D_0^3-D_1)(D_0^3-4D_1)f\cdot f=0$,

$D_0(D_0^3-D_1)(D_0^3+2D_1)(D_0^3-4D_1)f\cdot f=0$, $(D_0^3-D_1)(D_0^3-4D_1)(D_0^6+27D_0^3D_1+16D_1^2)f\cdot f=0$

weight 14 以上では $D_0, D_1$ のみの方程式は存在しない。

$D_0(D_0^3D_1^2+D_1^3-2D_4)f\cdot f=0$, $(2D_0D_1^3+3D_2^2-3D_1D_3-2D_0D_4)f\cdot f=0$,

$D_1(2D_0^6D_1+3D_0D_1D_2-5D_4)f\cdot f=0$, $(D_0^4D_1D_2-D_0D_1^2D_2+D_0^5D_3-D_1D_4)f\cdot f=0$,

$(D_0D_1D_2^2-D_0^4D_1D_3+D_2D_4-D_1D_5)f\cdot f=0$, $(D_0^3D_1^2D_2-D_2D_4-3D_0^3D_5+3D_0D_6)f\cdot f=0$,

$D_0^2(D_1D_m-D_0D_{m+1})f\cdot f=0$, $(2D_0^3D_m+D_1D_m-3D_0D_{m+1})f\cdot f=0$,

$(2D_0^5D_m+3D_2D_m-5D_1D_{m+1})f\cdot f=0$　($m=0,1,2,\cdots$ ：安定性の系).

(表6)　　　MKdV 方程式

$D_0^2f\cdot g=0$, $(D_0^3-D_1)f\cdot g=0$, $D_0(D_0^3+2D_1)f\cdot g=0$, $D_0^2(D_0^3-D_1)f\cdot g=0$,

$D_0^3(D_0^3-4D_1)f\cdot g=0$, $D_1(2D_0^3+D_1)f\cdot g=0$, $D_0^4(D_0^3-D_1)f\cdot g=0$,

$D_0D_1(D_0^3-D_1)f\cdot g=0$, $D_0^2(D_0^3+2D_1)(D_0^3-4D_1)f\cdot g=0$, $D_0^3(D_0^3-D_1)(D_0^3+8D_1)f\cdot g=0$,

$D_1(D_0^3-D_1)^2f\cdot g=0$, $D_0^4(D_0^3-4D_1)^2f\cdot g=0$, $D_0D_1(2D_0^3+D_1)(D_0^3-4D_1)f\cdot g=0$,

$D_0^2(D_0^3-D_1)(D_0^6-32D_0^3D_1-32D_1^2)f\cdot g=0$, $D_0^2D_1(D_0^3-D_1)(D_0^3+2D_1)f\cdot g=0$,

$D_0^3(D_0^3+2D_1)(D_0^3-4D_1)^2f\cdot g=0$, $D_1(D_0^3+2D_1)(D_0^3-4D_1)(2D_0^3+D_1)f\cdot g=0$,

$D_0^4(D_0^3-D_1)(D_0^6-8D_0^3D_1-56D_1^2)f\cdot g=0$, $D_0D_1(D_0^3-D_1)(D_0^6+D_0^3D_1+16D_1^2)f\cdot g=0$,

$D_0^5(D_0^3-4D_1)^3f\cdot g=0$, $D_0^2D_1(D_0^3-4D_1)^2(2D_0^3+D_1)f\cdot g=0$, $\cdots\cdots$

weight 20 以上では $D_0, D_1$ のみの方程式は存在しない.

$(D_0^5-D_2)f\cdot g=0$ (Sym の例), $D_0(D_0^2D_1+D_2)f\cdot g=0$, $(D_0^7-D_3)f\cdot g=0$,

$D_0(D_0D_1^2+2D_0^2D_2+2D_3)f\cdot g=0$, $D_0^2(D_0D_1^2-D_3)f\cdot g=0$, $D_0(D_0^3D_1^2+2D_4)f\cdot g=0$,

$D_0^2(D_1^3-D_4)f\cdot g=0$, $D_0D_2(D_0^5-D_2)f\cdot g=0$, $D_0(D_0^3D_1D_2-D_1^2D_2-3D_0^2D_4)f\cdot g=0$,

$(D_0^2D_1^2D_2+D_1^2D_3-2D_0^4D_4)f\cdot g=0$, $D_0^2D_1(D_1^3-4D_4)f\cdot g=0$,

$(D_0^2D_m-D_{m+1})f\cdot g=0$, $\left(\sum_{\nu=0}^{m}(-)^\nu D_\nu D_{m-\nu}\right)f\cdot g=0$ (但し $m$ odd のときは trivial),
$\left(D_0^2\sum_{\nu=0}^{m}D_\nu D_{m-\nu}+2D_0D_{m+1}\right)f\cdot g=0$,
$(D_0^3D_m-D_1D_m+3D_0D_{m+1})f\cdot g=0$, $(D_0^4D_m+2D_0D_1D_m-3D_0^2D_{m+1})f\cdot g=0$,

$(D_0^5D_m-D_0^2D_1D_m-3D_0^3D_{m+1})f\cdot g=0$, $(D_0^5D_m-D_2D_m+5D_0D_{m+2})f\cdot g=0$,

$(D_0^7D_m-D_3D_m-7D_0^2D_1D_{m+1}+7D_0D_{m+3})f\cdot g=0$ ($m=0,1,2,\cdots$ : 安定性定理の系).

(表7) NLS 方程式

$(D_1^2-D_2)f\cdot g=0$, $D_1(D_1^2-D_2)f\cdot g=0$, $(D_1^2-D_2)(D_1^2+2D_2)f\cdot g=0$,

$D_1(D_1^2-D_2)(D_1^2-2D_2)f\cdot g=0$, $(D_1^2-D_2)(D_1^2-2D_2)^2f\cdot g=0$,

weight 7 以上では $D_1, D_2$ のみの方程式は存在しない。

$(D_1^3-D_3)f\cdot g=0$, $D_1(D_1^3+3D_1D_2-4D_3)f\cdot g=0$, $(D_1^4-3D_2^2+2D_1D_3)f\cdot g=0$,

$D_1(D_2^2-D_1D_3)f\cdot g=0$, $D_2(D_1^3-D_3)f\cdot g=0$, $(D_1^5+2D_1D_2^2-3D_2D_3)f\cdot g=0$,

$D_1(D_1^4+5D_1D_3-6D_4)f\cdot g=0$, $(D_1^2D_2^2-D_3^2+D_2D_4-D_6)f\cdot g=0$,

$(D_1D_2^3+3D_1^2D_2D_3-4D_7)f\cdot g=0$, $D_1(D_1^3D_3-D_3^2+3D_1D_5-3D_6)f\cdot g=0$,

$(D_1^3D_2D_3+D_2^2D_4-2D_8)f\cdot g=0$, $(D_1D_2^2D_3+D_2^2D_4-D_1^3D_5-D_1^2D_6)f\cdot g=0$,

$(D_1^8-5D_1^6D_2+20D_1^4D_4-16D_1^3D_5)f\cdot g=0$,

$(D_1D_m-D_{m+1})f\cdot g=0$, $(D_1^2D_m+D_2D_m-2D_1D_{m+1})f\cdot g=0$ ($m=1,2,3,\cdots$).

$D_1^2(D_1^4+3D_2^2-4D_1D_3)f\cdot f=0$, $(D_1^2D_2^2+2D_3^2-D_2D_4-2D_1D_5)f\cdot f=0$,

$(D_1^2D_2^2 + 2D_1^3D_3 + 5D_2D_4 - 8D_1D_5)\, f\cdot f = 0$,　$(D_1^2D_2D_3 + D_3D_4 - 2D_1D_6)\, f\cdot f = 0$,

$(D_1^2D_2D_3 + D_1^3D_4 + 3D_2D_5 - 5D_1D_6)\, f\cdot f = 0$,　$(D_1D_2^3 + 3D_3D_4 - 3D_2D_5 - D_1D_6)\, f\cdot f = 0$,

$(D_1^3D_m - D_3D_m + 3D_2D_{m+1} - 3D_1D_{m+2})\, f\cdot f = 0$　$(m=1, 2, 3, \cdots)$.

Jimbo - Miwa [8] の (31), (33) 式参照.

[付録 1]　Painlevé II 型方程式 : $y_{xx} = 2y^3 + xy + \alpha$, $y = (\log \frac{f}{g})_x$,

$$\begin{cases} D_x^2\, f\cdot g = 0 \\ (D_x^3 - xD_x - \alpha)\, f\cdot g = 0 \end{cases} \quad ([4]) \quad , \quad (D_x^6 - 4xD_x^4 - 4\alpha x D_x - 4\alpha^2)\, f\cdot g = 0.$$

[付録 2]　沢田 - 小寺 KdV 方程式 : $(D_x^6 - D_xD_t)\, f\cdot f = 0$, $D_x = D_0$, $D_t = D_2$

$D_0(D_0^5 - D_2)\, f\cdot f = 0$ ([2]),　$D_0(3D_0^7 + 7D_0^2D_2 - 10D_3)\, f\cdot f = 0$,

$(D_0^5 - D_2)(D_0^5 + 4D_2)\, f\cdot f = 0$,　$(6D_0^5D_2 - D_2^2 - 5D_0^3D_3)\, f\cdot f = 0$,

$D_0^3(D_0^7 - 21D_0^2D_2 + 20D_3)\, f\cdot f = 0$,　$(D_0^{12} + 7D_0^7D_2 + 7D_0^2D_2^2 - D_0^5D_3 - 14D_2D_3)\, f\cdot f = 0$.

[付録 3]　SIT 方程式 :

$$\begin{cases} D_x^2\, f\cdot g = 0 \cdots\cdots\cdots ① \\ (-D_t - D_x^2D_t + D_x)\, f\cdot g = 0 \cdots ② \end{cases} \quad \text{(Hirota, Oishi [4])}$$

$D_x = D_0$,　$D_t = \sum_{m=0}^{\infty} (-)^m D_m$　と取れば, ① のもとで, 1個の SIT 方程式② と 1系列の MKdV 方程式 : $(D_0^2D_m - D_{m+1})\, f\cdot g = 0$　$(m=0,1,2,\cdots)$　とは同値である。更に, 1個の SIT 方程式 : $(D_x^2D_t^2 - 2D_xD_t)\, f\cdot g = 0 \cdots ③$ と 1系列の MKdV 方程式 : $(D_0^2 \sum_{\nu=0}^{m} D_\nu D_{m-\nu} + 2D_0D_{m+1})\, f\cdot g = 0$　$(m=0,1,2,\cdots)$ も同値である。

(表8)

| $n$ | $p(n)$ | $p^{even}(n)$ | $q(n) = p^{odd}(n)$ | $q^{even}(n)$ | $q^{odd}(n)$ | $n$ | $p(n)$ | $p^{even}(n)$ | $q(n) = p^{odd}(n)$ | $q^{even}(n)$ | $q^{odd}(n)$ | $n$ | $p(n)$ | $p^{even}(n)$ | $q(n) = p^{odd}(n)$ | $q^{even}(n)$ | $q^{odd}(n)$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 1 | 1 | 20 | 627 | 42 | 64 | 10 | 7 | 40 | 37338 | 627 | 1113 | 64 | 46 |
| 1 | 1 | 0 | 1 | 0 | 1 | 21 | 792 | 0 | 76 | 0 | 8 | 41 | 44583 | 0 | 1260 | 0 | 49 |
| 2 | 2 | 1 | 1 | 1 | 0 | 22 | 1002 | 56 | 89 | 12 | 8 | 42 | 53174 | 792 | 1426 | 76 | 52 |
| 3 | 3 | 0 | 2 | 0 | 1 | 23 | 1255 | 0 | 104 | 0 | 9 | 43 | 63261 | 0 | 1610 | 0 | 57 |
| 4 | 5 | 2 | 2 | 1 | 1 | 24 | 1575 | 77 | 122 | 15 | 11 | 44 | 75175 | 1002 | 1816 | 89 | 63 |
| 5 | 7 | 0 | 3 | 0 | 1 | 25 | 1958 | 0 | 142 | 0 | 12 | 45 | 89134 | 0 | 2048 | 0 | 68 |
| 6 | 11 | 3 | 4 | 2 | 1 | 26 | 2436 | 101 | 165 | 18 | 12 | 46 | 105558 | 1255 | 2304 | 104 | 72 |
| 7 | 15 | 0 | 5 | 0 | 1 | 27 | 3010 | 0 | 192 | 0 | 14 | 47 | 124754 | 0 | 2590 | 0 | 78 |
| 8 | 22 | 5 | 6 | 2 | 2 | 28 | 3718 | 135 | 222 | 22 | 16 | 48 | 147273 | 1575 | 2910 | 122 | 87 |
| 9 | 30 | 0 | 8 | 0 | 2 | 29 | 4565 | 0 | 256 | 0 | 17 | 49 | 173525 | 0 | 3264 | 0 | 93 |
| 10 | 42 | 7 | 10 | 3 | 2 | 30 | 5604 | 176 | 296 | 27 | 18 | 50 | 204226 | 1958 | 3658 | 142 | 98 |
| 11 | 56 | 0 | 12 | 0 | 2 | 31 | 6842 | 0 | 340 | 0 | 20 | 51 | 239943 | 0 | 4097 | 0 | 107 |
| 12 | 77 | 11 | 15 | 4 | 3 | 32 | 8349 | 231 | 390 | 32 | 23 | 52 | 281589 | 2436 | 4582 | 165 | 117 |
| 13 | 101 | 0 | 18 | 0 | 3 | 33 | 10143 | 0 | 448 | 0 | 25 | 53 | 329931 | 0 | 5120 | 0 | 125 |
| 14 | 135 | 15 | 22 | 5 | 3 | 34 | 12310 | 297 | 512 | 38 | 26 | 54 | 386155 | 3010 | 5718 | 192 | 133 |
| 15 | 176 | 0 | 27 | 0 | 4 | 35 | 14883 | 0 | 585 | 0 | 29 | 55 | 451276 | 0 | 6378 | 0 | 144 |
| 16 | 231 | 22 | 32 | 6 | 5 | 36 | 17977 | 385 | 668 | 46 | 33 | 56 | 526823 | 3718 | 7108 | 222 | 157 |
| 17 | 297 | 0 | 38 | 0 | 5 | 37 | 21637 | 0 | 760 | 0 | 35 | 57 | 614154 | 0 | 7917 | 0 | 168 |
| 18 | 385 | 30 | 46 | 8 | 5 | 38 | 26015 | 490 | 864 | 54 | 37 | 58 | 715220 | 4565 | 8808 | 256 | 178 |
| 19 | 490 | 0 | 54 | 0 | 6 | 39 | 31185 | 0 | 982 | 0 | 41 | 59 | 831820 | 0 | 9792 | 0 | 192 |

# 参考文献


[1] G.H. Hardy and S. Ramanujan, Proc. London Math. Soc. 17 (1918).

[2] Y. Sawada and Y. Kotera, Progr. Theor. Phys. 51 (1974).

[3] R. Hirota, Lect. Notes in Math. 515, Springer (1976).

[4] S. Oishi, Generalized Soliton Solutions for Simultaneous Bilinear Soliton Equations, preprint, Waseda Univ. (1979).

[5] Y. Matsuno, Bilinearization of Nonlinear Evolution Equations, preprint, Kyoto Univ. (1980), to appear in J. Phys. Soc. Japan 48 (1980).

[6] Y. Matsuno, Bilinearization of Nonlinear Evolution Equations II. Higher-order modified Korteweg-de Vries Equations, preprint, Kyoto Univ. (1980), submitted to J. Phys. Soc. Japan.

[7] M. Jimbo and T. Miwa, Monodromy preserving deformation of linear ordinary differential equations with rational coefficients II, RIMS preprint 327 (1980).

[8] M. Jimbo and T. Miwa, Deformation of linear ordinary differential equations III, RIMS preprint 329 (1980).